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Systematic construction of hidden nonlocal symmetries for the inhomogeneous nonlinear diffusion equation

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Abstract
We consider a class of inhomogeneous nonlinear diffusion equations (INDE) that arise in solute transport theory. Hidden nonlocal symmetries that seem not to be recorded in the literature are systematically determined by considering an integrated equation, obtained using the general integral variable, rather than a system of first-order partial differential equations (PDEs) associated with the concentration and flux of a conservation law. Reductions for the INDE to ordinary differential equations (ODEs) are performed and some invariant solutions are constructed.

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1. Introduction

In modelling transport of adsorbing solutes in soils, the resulting diffusion–adsorption equation happens to be nonlinear of Fokker–Planck type. Using methods by Munier *et al* [10], power-law nonlinear adsorption–diffusion equations transform into a canonical form of the INDE:

$$f(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(g(x)u^n\frac{\partial u}{\partial x}\right). \tag{1}$$

Furthermore, one may adopt a new spatial variable

$$x' = \int_0^x f(s)\,\mathrm{d}s$$

so that it may be assumed, without loss of generality, that $f = 1$. Equation (1) is of fundamental importance in the diffusive transport processes that occur in mathematical physics. Naturally, this has provided motivation to find possible reductions to ordinary differential equations, along with special exact solutions. Potential symmetry analysis for this equation has been carried out by Khater *et al* [7] and Sophocleous [19–21].

We observe [5] that equation (1) with special case $n = -2$, $f(x) = 1$ and $g(x) = x^2$, is an integrable equation. It is transformable into the well-known potential-symmetry-admitting Fujita–Storm equation [6, 22]

$$\theta_t = [\theta^{-2}\theta_x]_x, \tag{2}$$

most simply [13] by the point transformation $\theta = u/x$. The Fujita–Storm equation (2) may be further simplified to the standard constant-coefficient linear diffusion equation by nonlocal transformations [22, 8, 4, 21]. Therefore, this case must give rise to potential symmetries. However, the classification reported in [7, 20] indicates no potential symmetries for this special case. The source of this discrepancy should be investigated so that the same problem can be avoided when other classes of partial differential equations are analysed.

It is common when constructing potential symmetries to first express the governing PDE in conserved form as a system of first-order PDEs. Some PDEs may be expressed as an auxiliary system in a number of inequivalent ways. For example [14] the linear wave equation

$$\frac{\partial^2 u}{\partial t^2} = x\frac{\partial^2 u}{\partial x^2} \tag{3}$$

may be expressed in two distinct ways as a system of first-order PDEs, namely

$$\frac{\partial \phi}{\partial x} = \frac{1}{x}\frac{\partial u}{\partial t}, \qquad \frac{\partial \phi}{\partial t} = \frac{\partial u}{\partial x}, \tag{4}$$

and

$$\frac{\partial \phi}{\partial x} = \frac{\partial u}{\partial t}, \qquad \frac{\partial \phi}{\partial t} = x\frac{\partial u}{\partial x} - u, \tag{5}$$

where ϕ is the potential variable. The first system indicates a globally conserved quantity

$$\int \frac{1}{x}u_t \, \mathrm{d}x,$$

with corresponding flux density $-u_x$. The second system indicates a globally conserved quantity

$$\int u_t \, \mathrm{d}x$$

with corresponding flux density $u - xu_x$. In this example, system (4) admits point symmetries that induce potential symmetries for the wave equation (3), whereas system (5) does not. This example shows that the choice of the specific form of the auxiliary system is crucial when making a symmetry classification. If we naively adopt standard integral variables such as ϕ of equation (5), then some important potential symmetries will remain hidden.

In section 2, we discuss modelling the transport of adsorbing solutes in soils. In section 3, we develop a systematic method that finds all potential symmetries, including those that are missed by standard potential symmetry analysis. In this method, we first assume a general weighted integral variable and then determine symmetries of an integrated form of the original governing equation, rather than a system of first-order PDEs associated with it. Lastly, in section 4 we present some reductions of the INDE by elements of the optimal system of one-parameter groups of potential symmetries.

2. Nonlinear adsorption–diffusion of solutes in porous media

Combining the equation of continuity for mass conservation

$$\frac{\partial(\theta(t,z)c)}{\partial t} + \frac{\partial J}{\partial z} = 0,$$

with the form of solute flux density (see, e.g., [1])

$$J = -\theta(t,z)D(v)\frac{\partial c_f}{\partial z} + qc_f$$

we obtain the adsorption–diffusion equation

$$\frac{\partial(\theta(t,z)c)}{\partial t} = \frac{\partial}{\partial z}\left[D(v)\theta(t,z)\frac{\partial c_f}{\partial z} - qc_f\right], \tag{6}$$

where $c = c_a + c_f$ is the total concentration of one chemical species, c_f is the concentration within the liquid solution, c_a is the concentration of the adsorbed component, q is the Darcian water flux, $\theta(t,z)$ is the volumetric water content, t is time, z is the vertical depth measured positively downwards and $D(v)$ is the pore water velocity-dependent dispersion coefficient. Experimental observations [1] reveal that $D(v)$ is approximately in linear proportion to the pore water velocity $v = \frac{|q|}{\theta}$. We assume $D(v) = \delta v$, where δ is the dispersion length. $D(v)$ includes both molecular and mechanical dispersion [1]. If the adsorption process is bimolecular and the desorption process is monomolecular (see [16]) then the equilibrium condition is

$$\frac{c_f c}{c_a} = \kappa. \tag{7}$$

Since $c = c_a + c_f$, the locally free concentration is given by

$$c_f = \frac{c}{1+\kappa^{-1}c}, \tag{8}$$

where κ is the equilibrium constant [16]. For steady water flow and using the equation for continuity, we obtain the nonlinear adsorption–diffusion equation

$$\frac{\theta(z)}{(1-\kappa^{-1}c_f)^2}\frac{\partial c_f}{\partial t} = \frac{\partial}{\partial z}\left(\delta|q|\frac{\partial c_f}{\partial z}\right) - q\frac{\partial c_f}{\partial z}. \tag{9}$$

With application to evaporation from a water table at a constant Darcian water flux $q = -R$, so that R is the evaporation rate, equation (9) then becomes

$$\frac{\theta(z)}{(1-\kappa^{-1}c_f)^2}\frac{\partial c_f}{\partial t} = \delta R\frac{\partial^2 c_f}{\partial z^2} + R\frac{\partial c_f}{\partial z}. \tag{10}$$

In terms of scaled dimensionless variables we write equation (10) as

$$\frac{\theta_*(z_*)}{(1-c_*)^2}\frac{\partial c_*}{\partial t_*} = \frac{\partial^2 c_*}{\partial z_*^2} + \frac{\partial c_*}{\partial z_*}. \tag{11}$$

Here $c_* = c_f/c_s$, $t_* = t/t_s$, $z_* = z/\lambda_s$, $\theta_* = \theta/\theta_s$, with c_s being a suitable concentration cale, $c_s = \kappa$, $\delta = \lambda_s$ and $t_s = \theta_s\delta/R$. θ_s is the water content at saturation and λ_s is a macroscopic sorptive length scale [23].

3. Classical potential symmetry analysis

Construction and applications of potential symmetries may be found in [2]. Let $\mu = 1 - c_*$, then under $\mu = \mathrm{e}^{-z_*} v(\hat{z}, t_*)$, $\hat{z} = \mathrm{e}^{z_*}$ [10], equation (11) transforms to

$$\frac{\partial v}{\partial t_*} = H(\hat{z}) v^2 \frac{\partial^2 v}{\partial \hat{z}^2}, \tag{12}$$

where $H(\hat{z}) = \frac{1}{\theta_*(z_*)}$. Equation (12) is linearizable only when $H(\hat{z})$ is quadratic in $\hat{z}$ [10]. The transformation

$$v^{-1} = u, \qquad y = \int \frac{1}{H(\hat{z})} \, \mathrm{d}\hat{z},$$

allows us to write equation (12) as a local conservation law

$$\frac{\partial u}{\partial t_*} = \frac{\partial}{\partial y}\left(u^{-2} G(y) \frac{\partial u}{\partial y}\right), \tag{13}$$

where $G(y) = \mathrm{d}y/\mathrm{d}\hat{z}$. Equation (13) may naturally be split into an auxiliary system

$$\frac{\partial \phi}{\partial y} = u, \qquad \frac{\partial \phi}{\partial t_*} = u^{-2} G(y) \frac{\partial u}{\partial y}, \tag{14}$$

where ϕ is the potential variable. Symmetry analysis of the auxiliary system (14) using Dimsym [17] reconfirms the results obtained for the cases $G =$ constant, and the power law $G(y) = y^4$ [2, 7, 18–20]. Note that potential symmetries are invisible for the system (14) with the power law (15). The second equation in the system (14) may further be split into another auxiliary system since it may be written in a conserved form (see, e.g., [20]). We refrain from expressing equation (13) as a system of first-order PDEs or auxiliary system (14), but rather show that in the undetected special case

$$G(y) = y^2, \tag{15}$$

equation (13) admits extra potential symmetries and that these may be constructed by considering the integrated form of the governing equation in terms of a general integral-dependent variable

$$\phi = \int k(y)\, u(y, t_*) \, \mathrm{d}y + J(t_*), \tag{16}$$

where $J(t_*)$ is a constant of integration. This ansatz provides a wider freedom in the definition of the integral variable, through the kernel k. The determining equations for non-generic potential symmetries now include differential equations for $k(x)$. Substituting (16) into (13) we obtain a third-order partial differential equation

$$\frac{\phi_{yt_*} \phi_y^2}{(yk)^2} = \frac{2}{yk}\left(1 - \frac{\phi_{yy}}{\phi_y}\right)(k\phi_{yy} - k'\phi_y) + \phi_{yyy} - \frac{k''\phi_y}{k}. \tag{17}$$

Classical point symmetries of equation (17) yield nonlocal or potential symmetries of equation (13) (see, e.g., [2]) provided the transformed points $(\bar{y}, \bar{t}, \bar{u})$ of graph space depend explicitly on ϕ. In the classical symmetry analysis of equation (17) with arbitrary k, Dimsym [17] reports that the admitted principal Lie algebra is spanned, in the generic case, by the base vectors

$$\Gamma_1 = \phi \frac{\partial}{\partial \phi} + 2t_* \frac{\partial}{\partial t_*}, \qquad \Gamma_2 = \frac{\partial}{\partial t_*} \qquad \text{and} \qquad \Gamma_{1\infty} = h(t_*) \frac{\partial}{\partial \phi}. \tag{18}$$

It remains to find all possible functions k for which equation (17) admits extra point symmetries. Dimsym [17] reports that extra symmetries may be obtained only if k is a solution of the first-order linear ODE

$$yk' + k = 0, \tag{19}$$

or the third-order nonlinear ODE

$$(y^3kk' + y^2k^2)k''' + (y^3(k')^2 - 2y^3kk'' - 3y^2kk' + 4yk^2)k'' + (2y^2(k')^2 - 4ykk' + 2k^2)k' = 0, \tag{20}$$

or the fourth-order nonlinear ODE

$$\begin{aligned}(y^5k^2(k')^2 + 2y^4k^3k' + y^3k^4)k^{(iv)} &+ (7y^2k^4 + 5y^3k^3k' + y^4k^2(k')^2 + 3y^5k(k')^3 - 6y^4k^3k'' \\ &- 6y^5k^2k'k'')k''' + (29y^3k^2(k')^2 - 25y^2k^3k' - 7y^4k(k')^3 + 10yk^4 \\ &+ y^5(k')^4 - 23y^3k^3k'' + 8y^4k^2k'k'' - 5y^5k(k')^2k'' + 6y^5k^2(k'')^2)k'' \\ &+ (2k^4 - 12yk^3k' + 20y^2k^2(k')^2 - 12y^3k(k')^3 + 2y^4(k')^4)k' = 0.\end{aligned} \tag{21}$$

The general solution to equation (19) is $k = a_1/y$. After setting the inconsequential non-zero free constant a_1 to unity, equation (13) integrates completely to

$$\phi_{t_*} = \frac{\phi_{yy}}{\phi_y^2} + w(t_*), \tag{22}$$

where $w(t_*)$ is a constant of integration which, without loss of generality, will herein be equated to zero. Dimsym [17] finds extra point symmetries admitted by equation (22) namely,

$$\begin{aligned}&\Gamma_3 = \frac{\partial}{\partial\phi}, \qquad \Gamma_4 = y\frac{\partial}{\partial y}, \qquad \Gamma_5 = 2t_*\frac{\partial}{\partial\phi} - \phi y\frac{\partial}{\partial y}, \\ &\Gamma_6 = 4\phi t_*\frac{\partial}{\partial\phi} - y(\phi^2 + 2t_*)\frac{\partial}{\partial y} + 4t_*^2\frac{\partial}{\partial t_*}, \qquad \text{and} \qquad \Gamma_{2\infty} = S(\phi, t_*)\frac{\partial}{\partial\phi},\end{aligned} \tag{23}$$

where S is a general solution of a linear diffusion equation

$$S_{\phi\phi} - S_{t_*} = 0,$$

indicating the linearization [3] of equation (22). Γ_5 and Γ_6 induce nonlocal symmetries admitted by equation (13). The special choice $k = 1/y$ in (16) is associated with the fact that equation (22) with $w(t_*) = 0$ can be expressed as an alternative conservation law, equivalent to the auxiliary system

$$\phi_y = \frac{u}{y}, \qquad \phi_{t_*} = \frac{y}{u^2}\frac{\partial u}{\partial y} - \frac{1}{u}. \tag{24}$$

The corresponding globally conserved quantity is

$$\int \frac{u}{y}\,\mathrm{d}y$$

with flux

$$\frac{1}{u} - \frac{y}{u^2}u_y.$$

We refer to (24) as a 'hidden auxiliary system'. This particular auxiliary system is in fact equivalent to the standard auxiliary system of (2) by the transformation [13]

$$\theta = u/y, \quad x = y.$$

However, by the general approach developed here, no prior knowledge of the equivalence class of the target PDE is necessary. It could be applied just as readily to non-integrable PDEs and the algorithm can easily be implemented using a symmetry-finding algorithm implemented on a computer algebra package. Equation (20) is equivalent to

$$[\log p''(y) - 2\log p'(y) + \log p]' = 0, \tag{25}$$

where $p = yk(y)$. Hence the general solution to (20) is

$$k = (a_1 + a_2 y)^{a_3}/y \tag{26}$$

or

$$k = a_1\,\mathrm{e}^{a_2 y}/y, \tag{27}$$

with a_i arbitrary constants. Each of these special choices for $k(y)$ leads to additional symmetries for (17). In the case of (26), the additional symmetries are generated by

$$\Gamma_1 = 2t\frac{\partial}{\partial t} - \frac{y}{a_3}\frac{\partial}{\partial y} - \frac{a_1}{a_2 a_3}\frac{\partial}{\partial y},$$

and

$$\Gamma_2 = \phi\frac{\partial}{\partial \phi} + \frac{y}{a_3}\frac{\partial}{\partial y} + \frac{a_1}{a_2 a_3}\frac{\partial}{\partial y}.$$

In the case of (27), the additional generators are

$$\Gamma_1 = 2t\frac{\partial}{\partial t} - \frac{1}{a_2}\frac{\partial}{\partial y}$$

and

$$\Gamma_2 = \phi\frac{\partial}{\partial \phi} + \frac{1}{a_2}\frac{\partial}{\partial y}.$$

However, these are merely extensions of point symmetries for (13). Both equations (20) and (21) admit the solutions $k = 1/y$ and $k =$ constant. Equation (21) is a differential consequence of equation (20) and we suspect that no further nonlocal symmetry-bearing cases of k exist.

Now applying this method to a more general PDE (13), where $G(y)$ is arbitrary and the term u^{-2} is generalized to u^n, we obtain results which are already given in [2, 7, 19, 20] and recover the results obtained in (18) and (23) for $G(y) = y^2$. No further new cases were obtained and we herein omit these calculations.

4. Potential symmetry reductions for the INDE

Using the method employed by [11], we find the one-dimensional Ovsiannikov [12] optimal system,

$$\{\Gamma_1 + a\Gamma_4, \pm\Gamma_2 \pm \Gamma_3 + \Gamma_6, \Gamma_2 + \Gamma_6, \Gamma_3 + \Gamma_6, \Gamma_6, \Gamma_4 \pm \Gamma_5, \Gamma_4\},$$

where a is an arbitrary constant. The canonical invariants and the reduced ODEs associated with each of these elements are given in table 1. Following reductions by all the elements of the optimal system except $\Gamma_4 + \Gamma_5$, the obtained solutions to equation (22) are given in implicit form. In the following, we construct a simple explicit invariant solution to equation (13).

Example (i). Reduction by $\Gamma_4 + \Gamma_5$ leads to a functional form

$$\ln y = \frac{1}{2}\left(\phi - \frac{\phi^2}{2}\right) + F(t_*),$$

where F satisfies the ODE

$$F'(t_*) = \frac{1}{4t_*^2} - \frac{1}{2t_*}.$$

Table 1. Nonlocal symmetry reductions for equation (22).

Symmetry	Reduced ODEs
$\Gamma_1 + a\Gamma_4$	$\phi = \sqrt{t_*}F(\rho)$, where $\rho = \frac{y}{\sqrt{t_*}}$ and F satisfies $F'' - \frac{1}{2}F(F')^2 + \frac{1}{2}\rho(F')^3 = 0$
$\Gamma_2 + \Gamma_3 + \Gamma_6$	$\ln y = -\frac{t_*(\phi - t_*)^2}{4t_*^2+1} - \frac{\phi}{2} + \frac{t_*}{4} + \frac{\arctan(2t_*)}{8} - \frac{\ln(4t_*^2+1)}{4} + F(\rho)$, where $\rho = \frac{\phi - t_*}{\sqrt{4t_*^2+1}}$ and F satisfies $F'' + (F')^2 + \rho^2 - \frac{1}{4} = 0$
$\Gamma_2 + \Gamma_6$	$\ln y = -\frac{\phi^2 t_*}{4t_*^2+1} - \frac{\ln(4t_*^2+1)}{4} + F(\rho)$, where $\rho = \frac{\phi}{\sqrt{4t_*^2+1}}$, and F satisfies $F'' + (F')^2 + \rho^2 = 0$
$\Gamma_3 + \Gamma_6$	$\ln y = -\frac{24t_*^2\phi^2+12t_*\phi+1}{96t_*^3} - \ln\sqrt{t_*} + F(\rho)$, where $\rho = \frac{8t_*\phi+1}{8t_*^2}$ and F satisfies $F'' + (F')^2 - \frac{\rho}{8} = 0$
Γ_6	$\ln y = -\frac{\phi^2}{4t_*} - \frac{\ln t_*}{2} + F\left(\frac{\phi}{t_*}\right)$, where F satisfies $F'' + (F')^2 = 0$

Thus,

$$\phi = a_2 \pm \sqrt{4a_1 t_* - 4t_* \ln y - 2t_* \ln t_*}, \tag{28}$$

is a solution to equation (22), with a_1 and a_2 arbitrary constants. Hence equation (13) admits a solution

$$u = \pm\sqrt{\frac{2t_*}{a_1 - 2\ln y - \ln t_*}}. \tag{29}$$

For t greater than zero, this solution may be extended continuously to $y = 0$ where it must satisfy the simple Dirichlet boundary condition $u = 0$. However, there is a singularity at the moving boundary

$$y = \mathrm{e}^{a_1/2} t_*^{-1/2}.$$

Similar solutions may be obtained when F = constant in the reduction by Γ_6, included in table 1. Solutions to related moving boundary problems on heterogeneous media were given in [15]. In a separate paper [9], we discuss associated solutions to the solute transport model outlined here in section 2.

5. Conclusion

Integral symmetries admitted by the INDE (13) have been obtained for the special case $G(y) = y^2$ by considering the integrated form expressed in terms of a general weighted integral-dependent variable. These symmetries are more general than those found by the standard algorithm for potential symmetries. Invariant solutions have been constructed using potential symmetries that are not found by the standard algorithm.

The use of a general weight function $k(y)$ in the definition (16) of the integral variable, produced an additional nonlocal symmetry only in the very special circumstance that the new weighted integral was a conserved quantity. For the particular class of PDE considered here, the subclass of equations possessing nonlocal symmetries is identical to the subclass of equations that can be expressed as conservation laws (Goard, private communication). We do not know if this observation, made without reference to variational formulations and Noether symmetries, is indicative of a more general principle.

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